PARAMETRIZATION OF RATIONAL MAPS ON A VARIETY OF GENERAL TYPE, AND THE FINITENESS THEOREM

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ABSTRACT. In a previous paper, we provided some update in the treatment of the finiteness theorem for rational maps of finite degree from a fixed variety to varieties of general type. In the present paper we present another improvement, introducing the natural parametrization of maps by means of the space of linear projections in a suitable projective space, and this leads to some new insight in the geometry of the finiteness theorem.

INTRODUCTION

Let X be an algebraic variety of general type, over the complex field. The dominant rational maps of finite degree $X \dashrightarrow Y$ to varieties of general type, up to birational isomorphisms $Y \dashrightarrow Y'$, form a finite set. We call this the *finiteness theorem for rational maps on a variety of general type.* The proof follows from the approach of Maehara [7] joined with some recent advances in the theory of pluricanonical maps, due to Hacon and McKernan [4] and to Takayama [8], [9].

In our paper [3], motivated by the wish of some effective estimate for the finite number of maps in the theorem, we provided some update and refinement in the treatment of the subject. We brought the rigidity theorem to a general form, avoiding certain technical restrictions, we pointed out the role of the canonical volume $\mathrm{vol}(K_X)$ in bounding the rational maps in the finiteness theorem, and we proposed a new argument leading to a refined version of the theorem.

However, something still not satisfactory was the use of a certain bunch of subvarieties of Chow varieties as a parameter space for rational maps, as in Maehara's approach is too. The most natural and simple parameter space should be the space of linear projections in a suitable projective space, already appearing for instance in the work of Kobayashi and Ochiai [5].

In the present paper we are able to replace the Chow parametrization with the natural parametrization, and this leads to some new insight into the geometry of the finiteness theorem. The main result concerns the structure of the special birational equivalence classes of maps viewed as unions of

connected components of a certain space of linear rational maps, see Theorem 6.1. This has as an immediate consequence a better refined finiteness theorem, see Theorem 6.2.

1. Preliminary material

a. Results on pluricanonical maps. A recent achievement in the theory of pluricanonical maps is the following theorem of uniform pluricanonical birational embedding, due to Hacon and McKernan [4] and to Takayama [8].

Theorem 1.1. *For any dimension n there is some positive integer r_n such that: for every n-dimensional variety V of general type the multicanonical divisor $r_n K_V$ defines a birational embedding $V \dashrightarrow V' \subset \mathbb{P}^M$.*

A basic tool is the canonical volume of a variety, the invariant arising in the asymptotic theory of divisors, see Lazarsfeld's book [6]. In terms of the canonical volume we have a bound

$$(1) \qquad\qquad \deg V' \leq \operatorname{vol}(r_n K_V),$$

see [4], Lemma 2.2. Moreover from elementary geometry we have a bound

$$(2) \qquad\qquad M \leq \deg V' + n - 1.$$

Note that the embedded variety V' needs not be smooth. Intimately related to the theorem above is the following result, proved in [4] and in [8].

Theorem 1.2. *For any dimension n there is some positive number ϵ_n such that every n-dimensional variety V of general type has $\operatorname{vol}(K_V) \geq \epsilon_n$.*

For instance, concerning the minimum r_n we know from the classical theory that $r_1 = 3$ and $r_2 = 5$, and a recent result is that $r_3 \leq 73$, while concerning the maximum ϵ_n it is clear that $\epsilon_1 = 2$ and $\epsilon_2 = 1$ and a recent result is $\epsilon_3 \geq 1/2660$, see J. A. Chen and M. Chen [1]. Note that [4] and [8] do not give explicit bounds for r_n and ϵ_n in the theorems above.

b. Bounds for the degree of a rational map. Let $f : X \dashrightarrow Y$ be a rational map of finite degree between varieties of general type. Because of Theorem 1.1, taking the r_n-canonical birational models X' and Y' in $\mathbb{P}^M$ (note that Y' lies within the embedding space of X'), the map f is identified with a *linear rational map* $X' \dashrightarrow Y'$, a rational map which is the restriction of a linear projection $\mathbb{P}^M \dashrightarrow \mathbb{P}^M$. For a linear map of finite degree the inequality $\deg f \deg Y' \leq \deg X'$ holds. Using (1) it follows that

$$(3) \qquad\qquad \deg f \leq \deg X' \leq (r_n)^n \operatorname{vol}(K_X).$$

A more precise estimate is as follows. For any rational map of finite degree the inequality $\deg f \, \mathrm{vol}(K_Y) \leq \mathrm{vol}(K_X)$ holds, see [3], Proposition 3.2. Using Theorem 1.2 it follows that

$$(4) \qquad \deg f \leq \frac{1}{\epsilon_n} \mathrm{vol}(K_X).$$

This bound is sharp for curves, and in this case it reduces to the usual bound from the Hurwitz formula.

c. Families of rational maps. Let T be a smooth variety. If $X \to T$ is a relative scheme over T, we denote by $X(t)$ the scheme fibre over t, and by X_t the associated reduced scheme.

A *family of varieties*, parametrized by a smooth variety T, is a surjective morphism $X \to T$, with X a variety, such that every scheme fibre $X(t)$ is: (i) irreducible, (ii) generically smooth (in order to be assigned multiplicity one in the associated algebraic cycle, see Fulton [2], Chap. 10), and (iii) of dimension equal to the relative dimension of X over T, of course. When the structure morphism is projective or smooth, we speak of a family of projective varieties or a family of smooth varieties.

A *family of rational maps* is the datum of a family of varieties $X \to T$ and a relative scheme $X' \to T$, over the same smooth variety T, and a rational map $f : X \dashrightarrow X'$, commuting with the structural projections, which for every $t \in T$ restricts to a rational map $f_t : X_t \dashrightarrow X'_t$.

d. The rigidity theorem. A family of rational maps *on a fixed variety* X is the datum of a relative scheme $Y \to T$, with T smooth, and a rational map

$$f : X \times T \dashrightarrow Y$$

which is a family of rational maps $f_t : X \dashrightarrow Y_t$ in the sense of the previous definition.

A *trivial family* is one which is obtained as follows. Let $h : X \dashrightarrow U$ be a rational map and let $g : T \times U \dashrightarrow Y$ be a birational isomorphism which is a family of birational isomorphisms $g_t : U \dashrightarrow Y_t$. Then the composite map

$$T \times X \xrightarrow{\;1 \times h\;} T \times U \xrightarrow{\;g\;} Y$$

is a trivial family, because all maps $g_t \circ h$ are birationally equivalent.

Recall that two dominant rational maps $f : X \dashrightarrow Y$ and $f' : X \dashrightarrow Y'$, defined on the same variety, are *birationally equivalent* if there is a birational isomorphism $g : Y \dashrightarrow Y'$ such that $f' = g \circ f$.

For projective varieties of general type and dominant rational maps of finite degree there are results of rigidity.

Theorem 1.3. *Let X be a smooth projective variety of general type. Let T be a smooth variety, let $Y \to T$ be a family of smooth projective varieties of general type, and let $f : X \times T \dashrightarrow Y$ be a family of rational maps of finite degree. Then f is a trivial family, so all maps f_t are birationally equivalent.*

The rigidity theorem above was proved by Maehara [7] with some technical restrictions, and has been brought to the present form in our previous paper [3], Theorem 2.1. More generally, if the family of image varieties is not known to be a smooth family, one has the following.

Corollary 1.4. *Let X be a projective variety of general type. Let T be a smooth variety, let $Y \to T$ be a family of projective varieties of general type, and let $f : X \times T \dashrightarrow Y$ be a family of rational maps of finite degree. There is a nonempty open subset T' of T such that the restriction $f|_{T'} : X \times T' \dashrightarrow Y|_{T'}$ is a trivial family.*

2. Graphs and images in a family of maps

Let $f : X \dashrightarrow X'$ be a family of rational maps parametrized by a smooth variety T, as in §1.c. Consider the relative product $X \times_T X'$ and call p and p' the projections to X and X'. *Assume now that $X \to T$ is a projective morphism.* Thus p' is a closed map. Then define:

- Γ the closed graph of f in $X \times_T X'$,
- Y the closed image of X in X',
- C any closed subscheme of X such that $X \smallsetminus C \to T$ is surjective and f is a regular map $X \smallsetminus C \to Y$,
- E the inverse image of C in Γ.

Note that $p'(\Gamma) = Y$, as p' is a closed map.

A natural question is whether $\Gamma \to T$ is the family of closed graphs for the given family of maps, more precisely: whether $\Gamma \to T$ is a family of varieties, as in §1.c, and every reduced fibre Γ_t coincides with the *closed graph $\Gamma(f_t)$*. A related question is whether $Y \to T$ is the family of closed images $\overline{f_t(X_t)}$, that is: whether $Y \to T$ is a family of varieties and every reduced fibre Y_t coincides with the *closed image $\overline{f_t(X_t)}$*. The following equality of reduced schemes holds:

$$\Gamma_t = \Gamma(f_t) \cup E_t$$

and from this, applying p', a description of Y_t follows.

Proposition 2.1. *In the setting above, assume that T is a smooth curve. (1) There is a nonempty open subset T' of T such that $\Gamma|_{T'} \to T'$ is the family of closed graphs for the restricted family $f|_{T'}$. (2) There is a nonempty open subset T'' of T' such that moreover $Y|_{T''} \to T''$ is the family of closed images for the family $f|_{T''}$.*

Proof. We start with an easy remark. Let $V \to T$ be a surjective morphism of varieties, with irreducible fibres, all of the same dimension. Then there is a nonempty open subset T' of T such that the restriction $V|_{T'} \to T'$ is a family of varieties. Now we apply this to the relative varieties Γ and Y over the curve T. In order to prove the statement we only need to identify the reduced fibres Γ_t and Y_t for sufficiently general t. This is what we do in the following.

(1) First, we show that $\Gamma_t = \Gamma(f_t)$ holds for every t if $E \to T$ is a flat morphism. Recall that this happens if and only if every irreducible component of E dominates T.

Write $\dim X =: n + 1$. We have $\Gamma_t = \Gamma(f_t) \cup E_t$. Remark that $\dim E < n + 1$. Then $\dim E_t < n$ for every t, because of flatness. But all components of Γ_t must have dimension $= n$ for every t. Thus E_t is not a component and $\Gamma_t = \Gamma(f_t)$, for every t. In particular, every Γ_t is irreducible of dimension n.

In the present situation, the statement follows from the remark in the beginning. In the general case, by generic flatness, we have that $E|_{T'} \to T'$ is flat for some T' and then, because of the remark, the statement follows.

(2) We know that $Y_t = p'(\Gamma_t)$, and for $t \in T'$ we have from (1) that $\Gamma_t = \Gamma(f_t)$ and hence $Y_t = \overline{f_t(X_t)}$. In particular every such Y_t is irreducible, and necessarily of dimension $= \dim Y - 1$. Because of the remark above, the statement follows. $\qquad\square$

In general, the family of graphs needs not exist for the full family of maps, as is seen later on in Remark 5.1.

3. The varieties of general type in a family

Using the technique of extension of differentials, from a special fibre to the total space of the family, we gave in [3], §1.4, a proof of the assertion that the property of being a variety of general type is invariant in a 1-dimensional small deformation, where small refers to the Zariski topology. Here we point out that the same proof shows indeed a slightly stronger assertion, to the effect that the same property 'propagates' from a component of a fibre.

Theorem 3.1. *Let T be a smooth irreducible curve, let Y be a variety and let $Y \to T$ be a projective morphism. Assume that some fibre Y_a has an irreducible component Z which is a variety of general type, and that the restriction $Y \smallsetminus Y_a \to T \smallsetminus \{a\}$ is a family of varieties, as in §1.c. Then there is a nonempty open subset T' of T such that Y_t is a variety of general type for $t \in T'$.*

Proof. Let $V \to Y$ be a resolution of singularities such that the strict transform Z' of Z is smooth. So Z' is of general type, and $\dim H^0(Z', mK_{Z'}) \geq cm^n$ for $m \gg 0$. Denote by π the composite map $V \to Y \to T$. Since $V \to T$ is generically smooth, and since $Y \to T$ is generically a family of varieties, restricting to some neighborhood of a, we may assume that for every $t \neq a$ the induced map $V_t \to Y_t$ is a resolution of singularities. As the general V_t is irreducible, it follows that every V_t is connected, by the Zariski connectedness theorem.

The extension theorem of Takayama [9] applies, and gives us that there is a surjective restriction homomorphism

$$\pi_* \mathcal{O}_V(mK_V) \otimes k(a) \quad \longrightarrow \quad H^0(Z', mK_{Z'}) \ .$$

The image $\pi_* \mathcal{O}_V(mK_V)$ is a torsion free coherent sheaf on the smooth curve T, hence it is a locally free sheaf. So the dimension of $\pi_* \mathcal{O}_V(mK_V) \otimes$

$k(t)$ is constant. For $t = a$ this dimension is $\geq cm^n$ for $m \gg 0$, by what we have seen above.

For $t \neq a$, since $mK_V|_{V_t} = mK_{V_t}$, one has the restriction homomorphism

$$\pi_* \mathcal{O}_V(mK_V) \otimes k(t) \longrightarrow H^0(V_t, \mathcal{O}_{V_t}(mK_V|_{V_t})) = H^0(V_t, mK_{V_t})$$

and in a smaller neighborhood of a we may assume that this is an isomorphism for $t \neq a$. It follows that $\dim H^0(V_t, mK_{V_t}) \geq cm^n$ for $m \gg 0$, hence Y_t is of general type. This holds for every t in a neighborhood of a. $\qquad \square$

4. Rigidity and limits

Another key point in our treatment is a result about limit maps in a generically trivial family of maps. The result that we give here is only slightly more general than the one in our previous paper, and the proof given here is more apparent.

Let X be a projective variety. Let T be a smooth irreducible curve, let $Y \to T$ be a projective morphism, and let $f : T \times X \dashrightarrow Y$ be a family of rational maps on X, as in §1.d. Assume that for every $t \in T$ the rational map $f_t : X \dashrightarrow \overline{f_t(X)}$ is of finite degree k.

Assume moreover that the family is *generically trivial*, as in Corollary 1.4, i.e. that there is a nonempty open subset T' of T such that the restriction $f|_{T'}$ is obtained as

$$T' \times X \xrightarrow{1 \times h} T' \times U \xrightarrow{g} Y|_{T'}$$

where $h : X \dashrightarrow U$ is a fixed dominant rational map, and where g is a birational isomorphism which restricts to a birational isomorphism $g_t : U \dashrightarrow Y_t$ for every $t \in T'$. Then $f_t = g_t \circ h$ for $t \in T'$, so all these maps are birationally equivalent, of degree $\deg(f_t) = k = \deg(h)$.

Proposition 4.1. *Assume that $f : T \times X \dashrightarrow Y$ is a family of rational maps of constant degree $\deg(f_t) = k$, and assume that the family is generically trivial, as in the setting above. Then all maps f_t are in the same birational equivalence class.*

Proof. Let $a \in T$ be any point, and let us prove that f_a is in the birational equivalence class of every f_t with $t \in T'$.

We may assume that U is a normal variety. Recall that for a rational map of varieties over a base curve, from a normal variety to a variety which is proper over the base, the exceptional locus is of codimension ≥ 2, by the valuative criterion of properness for instance. It follows that $g : T \times U \dashrightarrow Y$ restricts to a rational map $g_a : U \dashrightarrow Y_a$.

Since $f = g \circ (1 \times h)$ holds as an equality of rational maps $T \times X \dashrightarrow Y$ then there is equality of restrictions $f_a = g_a \circ h$. And since $\deg(f_a) = k = \deg(h)$ then $\deg(g_a) = 1$ and f_a is birationally equivalent to h and to every f_t. $\qquad \square$

5. Linear rational maps

Let $\mathbb{P}^m = \mathrm{P}(V^{m+1})$ and let $X \subseteq \mathbb{P}^m$ be a non degenerate subvariety, of dimension n. The space of linear maps $\mathbb{P}^m \dashrightarrow \mathbb{P}^m$ is the projective space
$$\mathbb{P}^N = \mathrm{P}(\mathrm{End}(V)) \quad \text{with } N = (m+1)^2 - 1.$$
We denote by $\alpha = \bar{\ell}$ a point in $\mathbb{P}^N$ and by $x = \bar{v}$ a point in $\mathbb{P}^m$. The evaluation homomorphism $(\ell, v) \mapsto \ell(v)$ determines a rational map
$$\mathbb{P}^N \times X \dashrightarrow \mathbb{P}^m$$
and this is the family of linear rational maps $\alpha : X \dashrightarrow \mathbb{P}^m$. We denote by $\overline{\alpha(X)}$ the closed image and by $\Gamma(\alpha)$ the closed graph of the map α.

The subscheme $C \subset \mathbb{P}^N \times X$ defined by $\ell(v) = 0$ is the exceptional locus of the rational map above. Consider the projection $C \to \mathbb{P}^N$. The fibre C_α is the trace in X of the center of the linear projection $\alpha : \mathbb{P}^m \dashrightarrow \mathbb{P}^m$.

Remark 5.1. The subscheme $\Gamma \subset \mathbb{P}^N \times X \times \mathbb{P}^m$ defined by $\ell(v) \wedge w = 0$ is the closed graph of the rational map above. Clearly Γ contains $C \times \mathbb{P}^m$. The projection $\Gamma \to \mathbb{P}^N$ does not define the family of graphs. The fibre is given by $\Gamma_\alpha = \Gamma(\alpha) \cup C_\alpha \times \mathbb{P}^m$. It is clear, just looking at dimensions, that $\Gamma_\alpha = \Gamma(\alpha)$ if and only if $C_\alpha = \emptyset$.

In $\mathbb{P}^N$ define the following subsets:

R the subset of all α such that $\alpha : X \dashrightarrow \overline{\alpha(X)}$ is of finite degree,
R_k the subset of all $\alpha \in R$ with $\deg(\alpha) = k$,

for every integer $k > 0$.

Proposition 5.2. (1) R is an open subset. (2) R_k is a constructible subset for every $k > 0$.

Proof. (1) In $(\mathbb{P}^N \times X) \setminus C$ let U be the subset of pairs (α, x) such that $\dim_x \alpha^{-1}(\overline{\alpha(X)}) = 0$. It is an open subset. In $\mathbb{P}^N$ the image of U coincides with R. In fact, if α admits some point $x \in X \setminus C_\alpha$ which is isolated in its fibre, then its general fibre is of dimension 0. As the projection $\mathbb{P}^N \times X \to \mathbb{P}^N$ is an open map, R is open in $\mathbb{P}^N$. (2) In $\mathbb{P}^N \times X^{\times k}$ let U_k be the subset of sequences $(\alpha, x_1, \ldots, x_k) =: (\alpha, \bar{x})$ such that every (α, x_i) belongs to U and $\alpha(x_1) = \cdots = \alpha(x_k)$ while in the sequence $(x_1, \ldots, x_k)$ there is no coincidence. For every $\alpha \in R$ denote by $U_k(\alpha)$ the fibre of U_k over α. Let V_k be the subset such that $\dim_{(\alpha, \bar{x})} U_k(\alpha) = n$. This is a locally closed subset in $\mathbb{P}^N \times X^{\times k}$. In $\mathbb{P}^N$ the image $V_k{}'$ of V_k is the locus of $\alpha \in R$ with $\deg \alpha \geq k$. In fact, if α admits some sequence $(x_1, \ldots, x_k)$ such that $\dim_{(\bar{x})} U_k(\alpha) = n$, as the projection $U_k(\alpha) \to X$ has 0-dimensional fibres, then $U_k(\alpha)$ dominates X, and hence for a general point x_1 the fibre of α contains at least k distinct points $x_1, \ldots, x_k$. It follows that R_k coincides with $V_k{}' \setminus V_{k+1}{}'$. $\square$

6. Refined finiteness theorem

Let X be a smooth projective variety of general type, of dimension n. Let $X' \subset \mathbb{P}^M$ be the image of X in the r_n-canonical birational embedding, see

Theorem 1.1. Here $M = h^0(X, r_n K_X) - 1$ is bounded above in (2). Every rational map of finite degree $f : X \dashrightarrow Y$ to a smooth projective variety of general type, taking the r_n-canonical model $Y' \subset \mathbb{P}^M$, gives rise to a linear rational map $\alpha : X' \dashrightarrow \mathbb{P}^M$ with $\overline{\alpha(X')} = Y'$.

In this natural way the set of birational equivalence classes of rational maps of finite degree from X to varieties of general type is injected into the set of birational equivalence classes of linear rational maps of finite degree from X' to $\mathbb{P}^M$. Our main result is concerned with the geometric structure of these special equivalence classes.

Theorem 6.1. *Let X be a smooth projective variety of general type. A birational equivalence class of rational maps of degree k from X to smooth projective varieties of general type forms a union of connected components of R_k.*

Proof. Let $\alpha \in R_k$ be such that $\overline{\alpha(X')}$ is of general type. Let T be a smooth irreducible curve with a morphism $T \to R_k$, that we write as $t \mapsto \alpha_t$, and with some point $a \in T$ such that $a \mapsto \alpha$. We claim that all maps α_t are birationally isomorphic to α.

Consider the rational map $T \times X' \dashrightarrow T \times \mathbb{P}^M$ which represents the family of maps α_t. Let Y be its closed image in $T \times \mathbb{P}^M$. There is a nonempty open subset T' of T such that $Y|_{T'} \to T'$ is the family of closed images, by Proposition 2.1.

The fibre Y_a contains $\overline{\alpha(X')}$, a variety of general type. It follows from Theorem 3.1 that, shrinking T' if necessary, we may assume that for every $t \in T'$ the variety $\overline{\alpha_t(X')}$ is of general type.

Then it follows from Corollary 1.4 to the rigidity theorem that, shrinking T' again, we may assume that the restriction $T' \times X' \dashrightarrow Y|_{T'}$ is a trivial family. And then it follows from Proposition 4.1 that all maps α_t with $t \in T$ are birationally equivalent, as we claimed.

So we reach the conclusion. Every irreducible curve through α in R_k is the image of a smooth irreducible curve T as above, and therefore is fully contained in the birational equivalence class of α. Therefore every connected curve through α in R_k is fully contained in the birational equivalence class of α. Since R_k is constructible, by Proposition 5.2, this means that the connected component of α in R_k is contained in the birational equivalence class of α. $\square$

The space R admits the stratification $\bigsqcup R_k$, where the degree k is bounded above in (3) in terms of the function r_n, or in (4) in terms of the function ϵ_n. As an immediate consequence of the previous result we obtain the following refined version of the finiteness theorem, which improves our previous result [3], Theorem 4.3.

Theorem 6.2. *Let X be a smooth projective variety of general type. The number of birational equivalence classes of rational maps of finite degree*

from X to smooth projective varieties of general type is bounded above by the number of connected components of strata in the stratification $R = \bigsqcup R_k$.

We showed in [3] that the finite number of classes of maps in the finiteness theorem has an upper bound of the form $B(n, v)$ where $n = \dim(X)$ and $v = \mathrm{vol}(K_X)$, and that such a function B can be explicitely computed in terms of the function r_n. This is obtained by means of rather cumbersome computations with the complexity of a certain bunch of subvarieties of Chow varieties, that was used as a parameter space for rational maps. We believe that an analogous computation working with the much simpler parametrization that has been established in the present paper will lead to a simpler procedure and to a better result for the function B.

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